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ON THE EXISTENCE OF ADDITIONAL INTEGRALS OF THE EQUATIONS OF MOTION OF A MAGNETIZABLE SOLID IN AN IDEAL FLUID, IN THE PRESENCE OF A MAGNETIC FIELD*

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The equations of motion of a magnetic solid in an ideal fluid and in a homogeneous magnetic field are derived, their Hamiltonian structure are studied and four first integrals are determined. The four-parameter family of the cases of Liouville integrability is found, as well as some cases of the existence of particular integrals.

1. Let us solid move in an unbounded volume of an ideal fluid which is at rest at infinity. We introduce two orthogonal coordinate system: a moving system $Oxyz$ rigidly bound to the solid, and a fixed system. We denote by $\mathbf{\Omega} = (\Omega^1, \Omega^2, \Omega^3)$ and $\mathbf{u} = (u^1, u^2, u^3)$ the vectors of instantaneous and translational velocity of the body. Here and henceforth the vector components are taken in the moving coordinate system. The kinetic energy of the "body + fluid" system is determined by the positive definite quadratic form

$$E(\mathbf{\Omega}, \mathbf{u}) = {}^1/_2\,\alpha_{ij}\Omega^i\Omega^j + \beta_{ij}\Omega^i u^j + {}^1/_2\,\gamma_{ij}u^i u^j$$

(the repeated indices everywhere denote summation from 1 to 3).

We introduce the vectors $\mathbf{M} = (M^1, M^2, M^3)$ and $\mathbf{p} = (p^1, p^2, p^3)$, $M^i = \partial E/\partial\Omega^i$, $p^i = \partial E/\partial u^i$, which can be regarded as the kinetic moment of the system about the point O and the total momentum. We denote by

$$H_0(\mathbf{M}, \mathbf{p}) = {}^1/_2\,a_{ij}M^iM^j + b_{ij}M^ip^j + {}^1/_2\,c_{ij}p^ip^j$$

the quadratic form dual to $E(\mathbf{\Omega}, \mathbf{u})$ relative to the Legendre transformation. Then

$$u^i = \partial H_0/\partial p^i,\ \Omega^i = \partial H_0/\partial M^i\ (i = 1, 2, 3) \tag{1.1}$$

The equations of the change of momentum and kinetic moment of the "body + fluid" system have the form /1/

$$\mathbf{p}^\cdot = \mathbf{p}\times\mathbf{\Omega} + \mathbf{F}_c,\quad \mathbf{M}^\cdot = \mathbf{M}\times\mathbf{\Omega} + \mathbf{p}\times\mathbf{u} + \mathbf{M}_c \tag{1.2}$$

where $\mathbf{E}_c$ and $\mathbf{M}_c$ are the additional non-hydrodynamic force and moment of forces acting on the body. The derivatives $\mathbf{M}^\cdot$ and $\mathbf{p}^\cdot$ determine the variation of the vectors $\mathbf{M}$ and $\mathbf{p}$ with respect to the moving coordinate system.

Let the body move in a homogeneous magnetic field $\mathbf{h}$. The magnetic field strength $\mathbf{h}^*$ and induction $\mathbf{b}^*$ distorted by the presence of the body satisfy, in the quasistatic approximation, satisfy the following equations and boundary conditions:

$$\operatorname{div}\mathbf{b}^* = 0,\ \operatorname{rot}\mathbf{h}^* = 0 \tag{1.3}$$
$$[b_n^*] = 0,\ [\mathbf{h}_\tau^*] = 0,\ \mathbf{h}^* \to \mathbf{h}\ \text{ as }\ x^2 + y^2 + z^2 \to \infty$$

where b_n^* and $\mathbf{h}_\tau^*$ are the normal component of the induction and tangential component of the magnetic field strength at the body surface. The square brackets denote the difference in the value of the quantity enclosed in them, on each side of the body surface.

If the body is linearly magnetizable, then $\mathbf{b}^* = \mu_1\mathbf{h}^*$ where μ_1 is the magnetic permeability of the material of the body. If the body is a permanent magnet, then $\mathbf{b}^* = \mathbf{h}^* + 4\pi\mathbf{\theta}$ where $\mathbf{\theta}$ is the constant magnetic dipole moment of the body. Let the relation connecting the induction and field strength within the body, have the form

$$\mathbf{b}^* = \mu_1\mathbf{h}^* + 4\pi\mathbf{\theta} \tag{1.4}$$

We shall assume that the fluid is linearly magnetizable and its magnetic permeability μ_2 is constant. Let us denote by Φ the free energy of the "body + fluid" system in the magnetic field. As we know (/2/, p. 170, $\mathbf{m}$ is the total dipole moment of the body)

$$\delta\Phi = -\langle\mathbf{m}, \delta\mathbf{h}\rangle,\quad \mathbf{m} = \frac{1}{4\pi}\int(\mathbf{b}^* - \mu_2\mathbf{h}^*)\,dV \tag{1.5}$$

The variation Φ is taken with the magnetic field sources constant. The angle brackets denote the scalar product of vectors.

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Since problem (1.3), (1.4) is linear, it follows that $\mathbf{m}$ depends linearly on $\mathbf{h}$, i.e.

$$\mathbf{m} = -(D\mathbf{h} + \mathbf{J}) \tag{1.6}$$

The matrix elements of the operator D and the components of the vector $\mathbf{J}$ are determined in the $Oxyz$ coordinate system by the geometry of the body and by the values of μ_1, μ_2, θ only. Integrating the relation (1.5) taking (1.6) into account, we find that $\Phi = \langle D\mathbf{h}, \mathbf{h}\rangle/2 + \langle \mathbf{J}, \mathbf{h}\rangle$ apart from an unimportant constant.

Varying Φ on the possible translations of the body, we obtain expressions for the force and moment of the force acting on the body from the side of the magnetic field (see e.g. /2/)

$$\mathbf{F}_c = 0, \quad \mathbf{M}_c = \mathbf{m} \times \mathbf{h} = -\partial\Phi/\partial\mathbf{h} \times \mathbf{h} \tag{1.7}$$

Introducing the notation

$$H(\mathbf{M}, \mathbf{p}, \mathbf{h}) = H_0(\mathbf{M}, \mathbf{p}) + \Phi(\mathbf{h}) = \langle A\mathbf{M}, \mathbf{M}\rangle/2 + \langle B\mathbf{M}, \mathbf{p}\rangle + \langle C\mathbf{p}, \mathbf{p}\rangle + \langle D\mathbf{h}, \mathbf{h}\rangle/2 + \langle \mathbf{J}, \mathbf{h}\rangle \tag{1.8}$$

we write (1.2), taking (1.1) and (1.7) into account, in the form

$$\mathbf{M}^{\cdot} = \mathbf{M} \times \partial H/\partial\mathbf{M} + \mathbf{p} \times \partial H/\partial\mathbf{p} + \mathbf{h} \times \partial H/\partial\mathbf{h} \tag{1.9}$$
$$\mathbf{p}^{\cdot} = \mathbf{p} \times \partial H/\partial\mathbf{M}$$

The equation

$$\mathbf{h}^{\cdot} = \mathbf{h} \times \partial H/\partial\mathbf{M} \tag{1.10}$$

describing the change in the value of the vector $\mathbf{h}$ in the moving coordinate system, makes it possible to obtain a closed system of equations in $\mathbf{M}, \mathbf{p}, \mathbf{h}$. Equations (1.9) and (1.10) together with the given function (1.8) form the subject of subsequent investigation.

The set of equations (1.9), (1.10) is also encountered in other problems of mechanics. Two examples of such problems follow.

1°. Let a polarizable, non-conducting solid move in an inbounded volume of an ideal incompressible fluid in the presence of a homogeneous magnetic field. In this case the equations are derived with the same accuracy as the equations of motion of a magnetic solid in a homogeneous magnetic field obtained above.

2°. Let us consider the motion of a satellite in a circular orbit, about its centre of mass /3/. Let $Oxyz$ be an orthogonal coordinate system rigidly bound to the satellite, with origin at the centre of mass, Λ be the inertial tensor of the satellite relative to the point O, ω_0 the angular velocity of motion of the point O along the circular orbit, $\boldsymbol{\omega}$ the absolute angular velocity of the satellite and $\mathbf{M} = \Lambda\boldsymbol{\omega}$ the kinetic moment vector. We introduce the function

$$H = \frac{1}{2}\langle \mathbf{M}, \Lambda^{-1}\mathbf{M}\rangle - \omega_0\langle \mathbf{M}, \mathbf{p}\rangle + \frac{3}{2}\omega_0^2\langle \mathbf{h}, \Lambda\mathbf{h}\rangle$$

Here $\mathbf{h}$ is the unit vector pointing from the centre of attraction towards the point O, and $\mathbf{p}$ is the unit vector normal to the orbital plane.

Then the motion of the satellite in the $Oxyz$ coordinate system will be described by equations (1.9), (1.10), with the function H of the type shown above.

2. Let us introduce into $C^\infty(R^9)$ the Poisson bracket, i.e. a bilinear skew symmetric operation $\{\cdot, \cdot\}$, satisfying the Leibnitz condition, assuming that

$$\{M_i, M_j\} = -e_{ijk}M_k, \quad \{M_i, p_j\} = -e_{ijk}p_k \tag{2.1}$$
$$\{M_i, h_j\} = -e_{ijk}h_k, \quad \{p_i, p_j\} = \{p_i, h_j\} = \{h_i, h_j\} = 0$$

The Eqs.(1.9),(1.10) can be written in the form

$$\mathbf{M}^{\cdot} = \{\mathbf{M}, H\}, \quad \mathbf{p}^{\cdot} = \{\mathbf{p}, H\}, \quad \mathbf{h}^{\cdot} = \{\mathbf{h}, H\} \tag{2.2}$$

System (2.2) represents a special case of Euler's equations in Lie algebra G /4/ consisting of the semi-direct sum of the Lie algebra of group E_3 of motions of three-dimensional Euclidean space, and Lie algebra of the group T_3 of translations of the three-dimensional space.

The system (1.9), (1.10)has four first integrals for any values of its parameters, namely the energy integral $I_1 = H$ and the integrals

$$I_2 = \langle \mathbf{p}, \mathbf{p}\rangle, \quad I_3 = \langle \mathbf{h}, \mathbf{h}\rangle, \quad I_4 = \langle \mathbf{p}, \mathbf{h}\rangle \tag{2.3}$$

The functions I_2, I_3, I_4 commute with any smooth functions on $R^9\{\mathbf{M}, \mathbf{p}, \mathbf{h}\}$, i.e. the Poisson bracket (2.1) is degenerate. Let us inspect the construction of the Poisson bracket and the function $H(\mathbf{M}, \mathbf{p}, \mathbf{h})$ on the non-singular level of the integrals

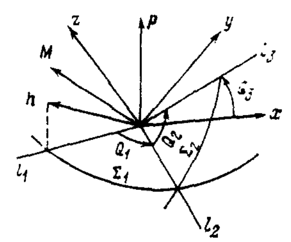
$$I_{234} = \{I_2 = c_2 > 0, I_3 = c_3 > 0, I_4 = c_4\}$$

Assertion 1. A global variable substitution exists on I_{234} transforming the Poisson bracket contracted on I_{234} to its canonical form.

Proof. We introduce the variables P_i, Q_i $(i = 1, 2, 3)$ on I_{234} as follows:

$$P_1 = M_3, \quad P_2 = \langle \mathbf{M}, \mathbf{M}\rangle^{1/2}, \quad P_3 = \langle \mathbf{M}, \mathbf{p}\rangle / \langle \mathbf{p}, \mathbf{p}\rangle^{1/2}$$

The meaning of the variables Q_i $(i = 1, 2, 3)$ is obvious from the figure.



Here $Oxyz$ is a moving orthogonal coordinate system rigidly bound to the body, Σ_1 and Σ_2 are planes passing through the point O and perpendicular to the vectors $\mathbf{p}$ and $\mathbf{M}$ respectively, l_1 is the orthogonal projection of the straight line passing along the vector $\mathbf{h}$ in the plane Σ_1; l_2 is the line of intersection of the planes Σ_1 and Σ_2; l_3 is the line of intersection of the planes Σ_2 and Oxy. The explicit expressions for the angles Q_i $(i = 1, 2, 3)$ written in terms of $\mathbf{M}, \mathbf{p}, \mathbf{h}$ are bulky, and are therefore not given here.

Direct computation of the Poisson bracket taking (2.1) into account, yields

$$\{P_i, Q_j\} = \delta_{ij}, \{P_i, P_j\} = \{Q_i, Q_j\} = 0$$

i.e. the variables P_i, Q_i $(i = 1, 2, 3)$ are canonical. The system of equations (2.2) on I_{234} has, in the new variables, the form

$$Q_i^{\cdot} = \partial H^* / \partial P_i, \; P_i^{\cdot} = -\partial H^* / \partial Q_i$$

where H^* is the contraction of the function H on I_{234}.

The variables P_i, Q_i $(i = 1, 2, 3)$ are analogous to the Andoyer variables used in investigating the dynamics of a heavy rigid body with a fixed point /5/.

3. The non-singular symplectic manifold I_{234} is six-dimensional, therefore the full integrability of the equations of motion (2.2) requires that, in addition to the energy integral, $I_1 = H$, another two first integrals exist commuting with each other.

Assertion 2. Let the function

$$H(\mathbf{M}, \mathbf{p}, \mathbf{h}) = \frac{1}{2}\langle A\mathbf{M}, \mathbf{M}\rangle + \frac{1}{2}\langle C\mathbf{p}, \mathbf{p}\rangle + \frac{1}{2}\langle D\mathbf{h}, \mathbf{h}\rangle$$

be such, that the matrices A, C, D are diagonal

$$A = \operatorname{diag}(a_1, a_2, a_3), \; C = \operatorname{diag}(c_1, c_2, c_3), \; D = \operatorname{diag}(d_1, d_2, d_3)$$

and

$$c_i = \varkappa_1 a_1 a_2 a_3 a_i^{-1} + \nu_1, \; d_i = \varkappa_2 a_1 a_2 a_3 a_i^{-1} + \nu_2 \tag{3.1}$$

where $\varkappa_1, \varkappa_2, \nu_1, \nu_2$ are arbitrary constants. Then system (2.2) has two additional first integrals commuting with each other

$$I_5 = \langle \mathbf{M}, \mathbf{M}\rangle - a_i(\varkappa_1 p_i^2 + \varkappa_2 h_i^2) \tag{3.2}$$
$$I_6 = \varkappa_1 \langle \mathbf{M}, \mathbf{p}\rangle^2 + \varkappa_2 \langle \mathbf{M}, \mathbf{h}\rangle^2 + \varkappa_1\varkappa_2 a_i (\mathbf{p} \times \mathbf{h})_i^2$$

and $I_1 = H, I_5, I_6$ are functionally independent on I_{234}.

Proof. Let us compute the derivatives $I_5^{\cdot}$ and $I_6^{\cdot}$ using system (2.2), with the function H of the type described above

$$I_5^{\cdot} = 2M_i M_i^{\cdot} - 2a_i(\varkappa_1 p_i p_i^{\cdot} + \varkappa_2 h_i h_i^{\cdot}) =$$
$$2e_{ijk}\{M_i(M_j a_k M_k + p_j c_k p_k + h_j d_k h_k) - a_i(\varkappa_1 p_i p_j + \varkappa_2 h_i h_j) a_k M_k\}$$

Reducing the similar terms we obtain

$$I_5^{\cdot} = 2e_{ijk} M_i p_j p_k (c_k + \varkappa_1 a_i a_k) + 2e_{ijk} M_i h_j h_k (d_k + \varkappa_2 a_i a_k)$$

By virtue of (3.1) the right-hand side of the last relation is zero. Similarly,

$$I_6^{\cdot} = 2\varkappa_1 \langle \mathbf{M}, \mathbf{p}\rangle \frac{d}{dt}\langle \mathbf{M}, \mathbf{p}\rangle + 2\varkappa_2 \langle \mathbf{M}, \mathbf{h}\rangle \frac{d}{dt}\langle \mathbf{M}, \mathbf{h}\rangle + 2\varkappa_1\varkappa_2 a_i s_i s_i^{\cdot} \tag{3.3}$$

where $\mathbf{s} = \mathbf{h} \times \mathbf{p}$. We have the relations

$$\frac{d}{dt}\langle \mathbf{M}, \mathbf{p}\rangle = e_{ijk} p_i h_j d_k h_k = -d_k s_k h_k$$
$$\frac{d}{dt}\langle \mathbf{M}, \mathbf{h}\rangle = e_{ijk} h_i p_j c_k p_k = c_k s_k p_k$$
$$a_i s_i s_i^{\cdot} = a_i s_i e_{ijk} s_j a_k M_k = \frac{a_1 a_2 a_3}{a_j} s_j (\mathbf{M} \times \mathbf{s})_j$$

Since $\mathbf{M} \times \mathbf{s} = \mathbf{M} \times (\mathbf{h} \times \mathbf{p}) = \mathbf{h}\langle \mathbf{M}, \mathbf{p}\rangle - \mathbf{p}\langle \mathbf{M}, \mathbf{h}\rangle$, it follows that $a_i s_i s_i^{\cdot} = a_1 a_2 a_3 a_k^{-1} s_k (h_k \langle \mathbf{M}, \mathbf{p}\rangle - p_k \langle \mathbf{M}, \mathbf{h}\rangle)$. Then, reducing the similar terms in (3.3) we obtain

$$I_6^{\cdot} = 2\langle \mathbf{M}, \mathbf{p}\rangle s_k h_k (-\varkappa_1 d_k + \varkappa_1\varkappa_2 a_1 a_2 a_3 a_k^{-1}) + 2\langle \mathbf{M}, \mathbf{h}\rangle s_k p_k (\varkappa_2 c_k - \varkappa_1\varkappa_2 a_1 a_2 a_3 a_k^{-1})$$

The right-hand side of this expression is zero, by virtue of (3.1) and the relations $\langle \mathbf{s}, \mathbf{p}\rangle = \langle \mathbf{s}, \mathbf{h}\rangle = 0$.

Thus we have a four-parameter family of Liouville-integrable systems of the form (1.9), (1.10). At the particular levels of I_{234} the integrals I_5 and I_6 are transformed, respectively, into a Klebsch integral and into the square of the area integral of the Kirchhoff equations /6/.

If $a_1 = a_2$, then the additional integrals can be taken in the form

$$I_5 = M_3$$
$$I_6 = \varkappa_1 \langle \mathbf{M}, \mathbf{p}\rangle^2 + \varkappa_2 \langle \mathbf{M}, \mathbf{h}\rangle^2 - \varkappa_1\varkappa_2 a_1 (p_1h_2 - p_2h_1)^2$$

In the case of system (2.2) with Hamiltonian H $(\mathbf{M}, \mathbf{p}, \mathbf{h}) = I_5$; the functions I_2, I_3, I_4, I_6 and

$$I = a_iM_i^2 + a_1a_2a_3a_i^{-1} (\varkappa_1p_i^2 + \varkappa_2h_i^2)$$

also form a complete set of independent commuting integrals.

We note that when the matrices A, B, C, D are diagonal and

$$a_1 = a_2,\ b_1 = b_2,\ c_1 = c_2,\ d_1 = d_2,\ J_1 = J_2$$

then $I_5 = M_3$ is the integral of system (2.2) with H of the form (1.8).

When the function H of the form (1.8) is chosen in a particular way, system (2.2) admits of a particular integral I, i.e. $I^{\cdot}|_{(2.2)} = 0$ if $I = 0$.

Assertion 3. Let the function H $(\mathbf{M}, \mathbf{p}, \mathbf{h})$ of the form (1.8) satisfy the conditions

$$A = \operatorname{diag}\ (a_1,\ a_2,\ a_3),\ a_1 < a_2 < a_3,\ \ B = 0,\ \ c_{12} = c_{23} = 0$$
$$\sqrt{a_2 - a_1}\ (c_{33} - c_{22}) \mp \sqrt{a_3 - a_2}c_{13} = 0$$
$$\sqrt{a_2 - a_1}c_{13} \pm \sqrt{a_3 - a_2}\ (c_{22} - c_{11}) = 0$$

and one of the following conditions:

$$1)\ J = 0,\ d_{12} = d_{23} = 0$$
$$\sqrt{a_2 - a_1}\ (d_{33} - d_{22}) \mp \sqrt{a_3 - a_2}d_{13} = 0$$
$$\sqrt{a_2 - a_1}c_{13} \pm \sqrt{a_3 - a_2}\ (c_{22} - c_{11}) = 0$$
$$2)\ D = 0,\ J_2 = 0$$
$$\sqrt{a_2 - a_1}J_3 \pm \sqrt{a_3 - a_2}J_1 = 0$$

Then $I = M_1\sqrt{a_2 - a_1} \pm M_3\sqrt{a_3 - a_2}$ is a particular integral of system (2.2).

The particular integral obtained is a generalization of the particular integral of the Kirchhoff equations /7/ and of the particular Hess-Appel'rot integral in the dynamics of a heavy rigid body with a fixed point.

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